Further questions

Countable subdirect powers of finite commutative semigroups

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23rd February 2022

(with kind support from CEMAT Ciencias.ID, ULisboa)

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Subdirect products

Definition

A subdirect product of two semigroups S and T is a subsemigroup U of the direct product $S \times T$ for which the projection maps

 $\pi_S: U \to S, \, (s,t) \mapsto s, \\ \pi_T: U \to T, \, (s,t) \mapsto t,$

onto S and T are surjections.

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Examples of subdirect products

- $\star\,$ The direct product $S\times T$ is a subdirect product of semigroups S and T.
- ★ $\Delta_S := \{(s,s) : s \in S\}$ is the diagonal subdirect product of a semigroup S with itself.
- \star Let F be the group with presentation

$$\left\langle x,y \mid [xy^{-1},x^{-1}yx] = [xy^{-1},x^{-2}yx^2] = 1 \right\rangle$$

Then $\langle (x, y^{-1}), (y, x), (x^{-1}, x^{-1}), (y^{-1}, y) \rangle$ is a subdirect product of F with itself, which is not equal to $F \times F$ or Δ_F .



Subdirect powers

Recall that for a family of sets $\{X_i\}_{i \in I}$ for some infinite indexing set I, the infinite Cartesian product is defined

$$\prod_{i \in I} X_i := \{ f : I \to \bigcup_{i \in I} X_i \mid (\forall i \in I) (f(i) \in X_i) \}.$$

It will be easier to view the elements of a countable Cartesian product of a family of sets as sets of countable strings in the following way;

$$\prod_{i\in\mathbb{N}} X_i = \{x_1 x_2 x_3 \dots : x_i \in X_i \text{ for } i\in\mathbb{N}\}.$$

The symbol x_1 will be called the "first component" of the string $x = x_1 x_2 x_3 \dots x_2$ will be the "second component", and so on. In general, the *i*-th component of a countable string x will be denoted $[x]_i$.

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Subdirect powers

If the sets X_i are all equal to the same set S, we will instead refer to the countable Cartesian product as a *countable Cartesian power*, denoted $S^{\mathbb{N}}$.

Exercise: A countable direct power of a semigroup S is a (possibly uncountably) infinite semigroup $S^{\mathbb{N}}$, with componentwise multiplication

$$s_1 s_2 s_3 \ldots \cdot t_1 t_2 t_3 \ldots = (s_1 \cdot t_1)(s_2 \cdot t_2)(s_3 \cdot t_3) \ldots$$

A subdirect power of a semigroup S is a subsemigroup of $S^{\mathbb{N}}$ for which the projection maps onto each component are surjections.



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A **countable** subdirect power of a semigroup S is a **countable** subsemigroup of $S^{\mathbb{N}}$ for which the projection maps onto each component are surjections.

Subdirect powers of finite groups



Theorem - McKenzie (1982)

A finite group G has countably many non-isomorphic countable subdirect powers if and only if G is abelian.

We'd like to work towards analogous results for subdirect powers of finite semigroups, that look like

Theorem

A finite semigroup S has countably many non-isomorphic countable subdirect powers if and only if S satisfies *insert fascinating semigroup properties here*.



Subdirect powers of finite groups

For this talk, we will concentrate on finite **commutative** semigroups.

Definition

A finite commutative semigroup S will be called *countable type* if it has only countably many non-isomorphic countable subdirect powers, and *uncountable type* if it has uncountably many such.



Some small examples

Firstly, the trivial semigroup of course is countable type, because $\{1\}^{\mathbb{N}}\cong\{1\}.$

The commutative semigroups of order 2 up to isomorphism are \mathbb{Z}_2 (countable type), O_2 (countable type) and $U_1 = \{0, 1\}$, the two element semilattice.

 U_1 can be viewed as a linearly ordered set with the natural ordering $0 \le 1$. Moreover, any subdirect power of U_1 will also be a semilattice, and can similarly be considered as an ordered set via $x \le y \iff x \land y = x$.



The case for U_1

A quick side definition:

Definition

For a finite word $u \in S^+,$ we will denote by \overline{u} the infinite string

$$uuu... \in S^{\mathbb{N}}.$$

An element v of $S^{\mathbb{N}}$ is said to be *recurring* if $v = \overline{u}$ for some finite word $u \in S^+$. Similarly, a subsemigroup of $S^{\mathbb{N}}$ is said to be recurring if all of its elements are recurring.

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The case for U_1

Lemma

For two recurring elements $s, t \in U_1^{\mathbb{N}}$ with $s \leq t$, there exists $u \in U_1^{\mathbb{N}}$ with $u \neq s$, $u \neq t$, but $s \leq u \leq t$.

Lemma

 $U_1^{\mathbb{N}}$ contains an order isomorphic copy of $\mathbb{Q},$ consisting of recurring elements .

Lemma

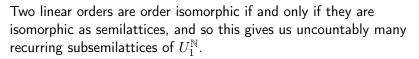
 $U_1^{\mathbb{N}}$ contains uncountably many linear orders consisting of recurring elements, up to order isomorphism.

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The case for U_1



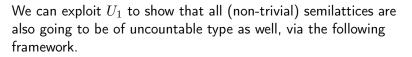
A subdirect product can be constructed from each one, and any two non-isomorphic semilattices will give non-isomorphic subdirect products with this construction.

This gives

Proposition U_1 is of uncountable type.

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Semilattices



If Y is a finite non-trivial semilattice with operation \wedge , then Y has a least element 0 and a minimal idempotent $e \neq 0$ with respect to the ordering on Y induced by \wedge .

 $\{0,e\}\cong U_1$, from which we can make uncountably many non-isomorphic countable recurring subdirect powers S. For any of these, it is easy to construct a countable recurring subdirect power of Y by "adding in the diagonal"

$$\overline{Y} = \{ \overline{y} \in Y^{\mathbb{N}} : y \in Y \}$$

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Semilattices

Two non-isomorphic recurring subdirect powers of U_1 will also give two non-isomorphic recurring subdirect powers of Y in this construction, and hence

Theorem

Any non-trivial semilattice Y is of uncountable type.

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Semigroups with more than one idempotent

Take a finite commutative semigroup S with E(S)>1, and let L be the greatest semilattice homomorphic image of S. Note that $L=S/\eta$, where η is the congruence on S defined

$$(a,b) \in \eta \Leftrightarrow (\exists x, y \in S^1)(\exists m, n \in \mathbb{N})(ax = b^m)(by = a^n).$$

Further, the congruence classes of η (that is, the elements of L) are precisely the Archimedean components of S.

Again, we can construct uncountably many recurring subdirect powers of L, from each of which we can construct a countable recurring subdirect power of S by making all possible "string replacements" by elements of the Archimedian component.

Semigroups with more than one idempotent

The greatest semilattice homomorphic image of the resulting subdirect power of S will be exactly the subdirect power of L you started with, which is an isomorphic invariant. Hence...

Theorem

Any finite commutative semigroup S wih E(S)>1 is of uncountable type.

Results 00000000●0000



Semigroups with a unique idempotent

Lemma

Let S be a finite commutative semigroup with a unique idempotent. Then S is either a group, or an ideal extension of a group by a k-nilpotent semigroup.

The case where S is a group has been dealt with. So it remains to consider ideal extensions of groups by k-nilpotent semigroups. We will start with k-nilpotent semigroups, noting that k = 2 has been dealt with.



k-nilpotent semigroups

Let S be a finite k-nilpotent commutative semigroup with $k \ge 3$, and let $M \subseteq \mathbb{N} \setminus \{1, 2, \dots, |S|\}$ be an infinite subset. Fix an element $x \in S^{k-1} \setminus \{0\}$ with d(x) minimal. Fix also an element $y \in S$ such that $y \mid x$.

Let

$$X = \left\{ (0)^{p-1} s \overline{0} \in S^{\mathbb{N}} : p \in \mathbb{N}, s \in S \right\},$$
$$Y_M = \bigcup_{p \in \mathbb{N}} \left\{ (0)^{p-1} y(x)^q \overline{0} \in S^{\mathbb{N}} : 1 \le q \le m_p \right\},$$

where m_p is the p-th element of M, when ordered in the natural way. Define

$$\mathcal{S}(S, x, y; M) := \langle X \cup Y_M \rangle \le S^{\mathbb{N}}.$$

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k-nilpotent semigroups

Lemma

For a k-nilpotent semigroup S with $k\geq 3,$ and $x,y\in S$ as above, if $M,N\subseteq\mathbb{N}\setminus\{1,\ldots,|S|\}$ with $M\neq N,$ then

 $\mathcal{S}(S,x,y;M) \not\cong \mathcal{S}(S,x,y;N)$

Corollary

Finite commutative k-nilpotent semigroups are of uncountable type.

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Extensions of *k*-nilpotent semigroups

The last cases to consider are ideal extensions of groups by k-nilpotent semigroups where the group is non-trivial, splitting into two cases: $k = 2, k \ge 3$.

There are similar complicated constructions for countable subdirect powers of these as above for the k-nilpotent case which I won't cover, but we have...

Corollary

Ideal extensions of non-trivial groups by k-nilpotent semigroups for $k \ge 2$ are of uncountable type.

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Final classification

Theorem (C, Ruškuc, 2021)

A finite commutative semigroup S is of countable type if and only if S is either a group, or a null semigroup.



Further questions

What are the types of non-commutative completely simple semigroups? What about finite semigroups in general?

Thank you for listening!